1 (a) $$y'' - 2y' - 8y = e^{4x}\cosh(x) \qquad (*)$$

$$m^2 - 2m - 8 = 0 : \quad (m-4)(m+2) = 0$$
$$\hookrightarrow m = 4, -2$$

$$y = A(x)e^{4x} + B(x)e^{-2x}$$
$$y' = A'e^{4x} + 4Ae^{4x} + B'e^{-2x} \cancel{+} -2Be^{-2x}$$

Set $A'e^{4x} + B'e^{-2x} = 0$. $\quad (2)$

$$y'' = 4A'e^{4x} + 16Ae^{4x} + 4Be^{-2x} - 2B'e^{-2x}$$

$(*)$: $$4A'e^{4x} + \cancel{16Ae^{4x}} + \cancel{4Be^{-2x}} - 2B'e^{-2x}$$
$$- \cancel{8Ae^{4x}} + \cancel{4Be^{-2x}} - \cancel{8Ae^{4x}} - \cancel{8Be^{-2x}} = e^{4x}\cosh(x)$$

$$4A'e^{4x} - 2B'e^{-2x} = e^{4x}\cosh(x) \qquad (1)$$
$$A'e^{4x} + B'e^{-2x} = 0 \qquad (2)$$

$(1)+2(2)$: $$6A'e^{4x} = e^{4x}\cosh(x)$$
$$6A' = \cosh(x) \quad \Rightarrow \quad A = \tfrac{1}{6}\sinh(x) + C$$

(2): $$B'e^{-2x} = -\tfrac{1}{6}\cosh(x)e^{4x}$$

$$B' = -\tfrac{1}{6}\cosh(x)e^{6x}$$
$$= -\tfrac{1}{6}\left[\frac{e^x + e^{-x}}{2}\right]e^{6x}$$
$$= -\tfrac{1}{12}\left[e^{7x} + e^{5x}\right]$$

$$\Rightarrow B = -\tfrac{1}{12}\left[\tfrac{1}{7}e^{7x} + \tfrac{1}{5}e^{5x}\right] + D$$

$$\Rightarrow y = \left[\tfrac{1}{6}\sinh(x) + C\right]e^{4x} \cancel{+} \left[-\tfrac{1}{12}\left\{\tfrac{1}{7}e^{7x} + \tfrac{1}{5}e^{5x}\right\} + \tilde{D}\right]e^{-2x}$$

can ignore constants.

(b) $\quad x y'' - (2x-1)y' - (8x+4)y = 0$

$$y(x) = \int_C e^{xt} f(t)\, dt$$

$$\rightarrow \int \left[x t^2 - (2x-1)t - (8x+4) \right] e^{xt} f(t)\, dt = 0$$

If this is $\int \frac{d}{dt}\left[e^{xt} g \right] dt$

$$\overset{\shortparallel}{e^{xt} g' + x e^{xt} g}$$

$$\Rightarrow \quad g' = (t-4) f \qquad g = (t^2 - 2t - 8) f$$
$$= (t-4)(t+2) f$$

$$\Rightarrow \quad \frac{g'}{g} = \frac{t-4}{t^2-2t-8} = \frac{t-4}{(t-4)(t+2)} = \frac{1}{t+2}$$

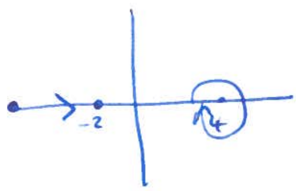
$$\Rightarrow \ln g = \ln(t+2) + \tilde{c}$$
$$g = C(t+2)$$
$$\rightarrow f = \frac{C}{t-4}$$

Choose C_i s.t. $\quad y(x) = \int \frac{C e^{xt}}{t-4} dt \neq 0$ $\quad t=4.$

and $\cancel{\left[e^{xt}(t+2) \right]_{C_i} = 0}$

$x = -\infty \qquad t = -2$

$\cancel{\text{and } \left[e^{xt}(t-4)(t+2) \right]}$



and $\left[C e^{xt}(t+2) \right] = 0$

$\uparrow \qquad \uparrow$

$x = -\infty \qquad t = -2$

$$y_1(x) = C \int_{-\infty}^{-2} \frac{e^{xt}}{t-4} \qquad \underline{\text{unbounded}}$$

$$y_2(x) = C \int_{C_2} \frac{e^{xt}}{t-4} dt = \underline{2\pi i\, e^{4x} C}$$

2 (a) $\frac{dy}{dx} = \frac{Cx+Dy}{Ax+By} = \frac{Q}{P}$

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$J|_{(0,0)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Check eigenvalues.

$$\begin{vmatrix} \lambda-A & -B \\ -C & \lambda-D \end{vmatrix} = (\lambda-A)(\lambda-D) - BC$$
$$\lambda^2 - D\lambda - A\lambda + AD - BC$$
$$\lambda^2 + (-D-A)\lambda + (AD-BC)$$
$$\lambda = \frac{D+A \pm \sqrt{(D+A)^2 - 4(AD-BC)}}{2}$$

Spiral if λ complex

Saddle if λ real but different signs

Node if λ real but same sign

Centre if λ imaginary.

−ve real part → stable
+ve real part → unstable

(b) In phase plane (xy), where $y = \dot{x}$.

A periodic trajectory is closed $\Longleftrightarrow$

$$x(t_0+T) = x(t_0) \qquad \forall t_0, \text{ for some } T$$
$$y(t_0+T) = y(t_0)$$

$\Rightarrow$ $x(t)$ is periodic with period T.

(c) $\ddot{x} = x(x^3 - \lambda^3)$

$$\rightarrow \left.\begin{aligned} \dot{y} &= x(x^3-\lambda^3) \\ \dot{x} &= y \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{x(x^3-\lambda^3)}{y} = \frac{Q}{P}$$

$-1(-1-7$

$$J = \begin{pmatrix} 0 & 1 \\ 4x^3-\lambda^3 & 0 \end{pmatrix}$$

$Q = 0 \Rightarrow x = 0$ or $x = \lambda$

$P = 0 \Rightarrow y = 0$

So singularities at $(0, 0)$
$(\lambda, 0)$.

$$J = \begin{pmatrix} 0 & 1 \\ 4x^3 - \lambda^3 & 0 \end{pmatrix}$$

$$J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -\lambda^3 & 0 \end{pmatrix}.$$ evals m:

$m^2 + \lambda^3 = 0$

$m^2 = -\lambda^3$

m complex, no real part.

<u>CENTRE.</u>

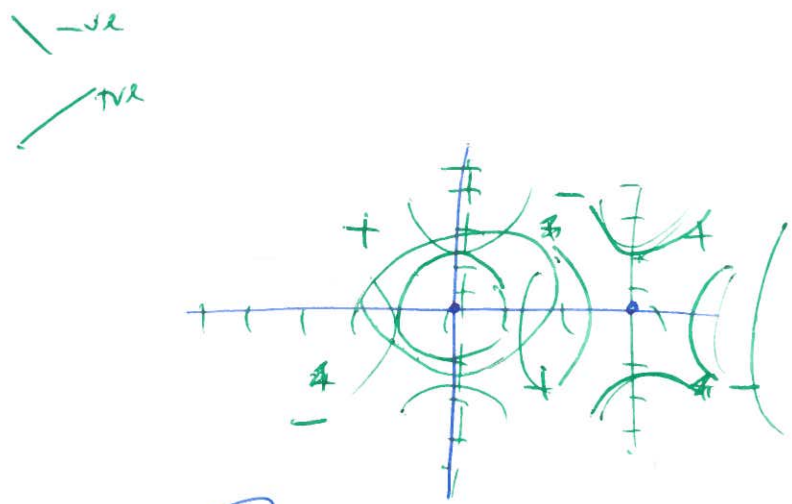
$$J|_{(\lambda,0)} = \begin{pmatrix} 0 & 1 \\ 3\lambda^3 & 0 \end{pmatrix}$$ evals m:

$m^2 - 3\lambda^3$

$m = \pm\sqrt{3\lambda^3}$

real, different sign

<u>SADDLE</u>



$y = U^2$ at $x=0$

Separatrices: at $(\lambda, 0)$, they have $y = bx$ so

$$\cancel{b} = \frac{dy}{dx} = \frac{x(x^3 - \lambda^3)}{y}$$

$\Rightarrow$ ~~b = ...~~ $\quad b = \frac{dy}{dx} = \frac{x(x^3 - \lambda^3)}{bx}$

Solve equation

$$\int y \, dy = \int x(x^3-\lambda^3)\, dx$$

$$\frac{1}{2} y^2 = \frac{1}{5}x^5 - \frac{1}{2}\lambda^3 x^2 + C$$

$y = U$ at $x=0$ $\Rightarrow$ $C = \frac{1}{2}U^2$

$$\Rightarrow y^2 = \frac{2}{5}x^5 - \lambda^3 x^2 + U^2$$

On the separatrix, at $\cancel{(0,\lambda)}$ $(\lambda, 0)$

$$0 = \frac{2}{5}\lambda^5 - \lambda^5 + U^2$$

$$\Rightarrow U^2 = \frac{3\lambda^5}{10} \quad \text{on separatrix}$$

Periodic motion happens "under" the separatrix

$$\Rightarrow U^2 < \frac{3\lambda^5}{10}.$$

$$T = \int dt = \int \frac{dt}{dx}\, dx = \boxed{\int \frac{dt}{dy}\frac{dy}{dx}\, dx}$$

$$= \cancel{\frac{x(x^3-\lambda^3)}{y}}$$

$$= \int y \, dx \rightarrow \int \left(\frac{2}{5}x^5 - \lambda^3x^2 + U^2\right)^{-1/2} dx.$$

3. $\ddot{x} + \varepsilon\dot{x}(4x-1) + x = 0.$

Let $\theta = nt \Rightarrow$

$\cancel{\frac{d}{d}}$ $\qquad \frac{d}{dt} = \frac{d}{d\theta}\frac{d\theta}{dt} = n\frac{d}{d\theta}$

$\Rightarrow \quad n^2x'' + \varepsilon n x'(4x-1) + x = 0 \qquad (*)$

$x(\theta)$ is 2π-periodic $\because$ In one period of t, (T), θ changes by 2π $(\therefore T = 2\pi/n)$.

$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

$n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots$

$(*)$: $[n_0 + \varepsilon n_1 + \varepsilon^2 n_2]^2[x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2''] + \varepsilon[n_0 + \varepsilon n_1 + \varepsilon^2 n_2][x_0' + \varepsilon x_1' + \varepsilon^2 x_2']$
$\cdot[4x_0 + 4\varepsilon x_1 + 4\varepsilon^2 x_2 - 1]$

$+ x_0 + \varepsilon x_1$ etc. CBA.

4. $\ddot{x} + \varepsilon f(x)\dot{x} + x = 0$ $\quad (*)$

Lienard transformation: $y = \frac{dx}{dt} + \varepsilon F(x)$

~~$\frac{dy}{dx} = \frac{d^2x}{dt^2} + \varepsilon F'(x)$~~

~~$\varepsilon f(x)$~~

$$\Rightarrow \quad \frac{dy}{dt} = \frac{d^2x}{dt^2} + \varepsilon F'(x)\dot{x}$$

$$= \ddot{x} + \varepsilon f(x)\dot{x}$$

$$\Rightarrow (*) \Rightarrow \frac{dy}{dt} + x = 0$$

$$\Rightarrow \frac{dy}{dt} = -x$$

$$\frac{dx}{dt} = \cancel{\varepsilon F}\, y - \varepsilon F$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{\varepsilon F - y}.$$

(b) "If a periodic sol^{n} exists then there is a unique closed ~~to~~ curve in the xy plane s.t.

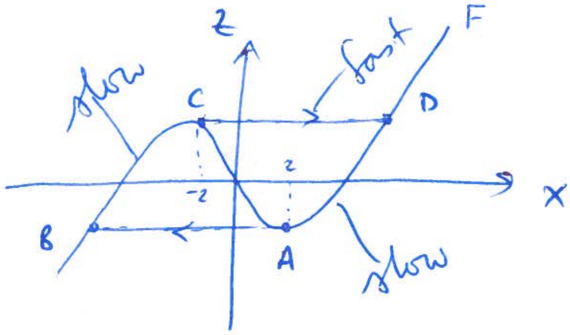
$$\oint_\gamma F(x)\,dy = 0$$

$$\rightarrow \oint \left(F(x)\frac{dy}{dx}\right)dx = 0$$

$$= \underbrace{\left[F(x)y\right]_\gamma}_{0} - \oint y F'(x)\,dx$$

$$\Rightarrow \oint y f(x)\,dx = 0.$$

(c) $f(x) = 2x^2 - 8.$ $\rightarrow$ $F(x) = \frac{2}{3}x^3 - 8x$



$$\Rightarrow \frac{dy}{dx} = \frac{x}{\epsilon\left[\frac{2}{3}x^3 - 8x\right] - y}$$

Let $y = \epsilon z$:

$$\Rightarrow \frac{dz}{dx} = \frac{x}{\epsilon^2\left[\frac{2}{3}x^3 - 8x - z\right]}$$

$$\Rightarrow \left(\frac{2}{3}x^3 - 8x - z\right)\frac{dz}{dx} = \frac{x}{\epsilon^2}$$

If $\epsilon \gg 1$, RHS is zero to leading order. We seek periodic sol^{n}s $\Rightarrow$ x, z remain finite since trajectory remains closed.

$$\Rightarrow \quad \frac{2}{3}x^3 - 8x - z = 0$$

$$\frac{dz}{dx} = 0$$

Find A: $F'(x) = 0$

$f(x) = 0$ ie $x^2 - 4 = 0 \Rightarrow x = \pm 2.$

$\left[C = (-2, \frac{32}{3})\right.$ by sym) $\Rightarrow F(2) = \frac{2}{3}8 - 16 = 16\left(-\frac{2}{3}\right) = -\frac{32}{3}$

$A = (2, -\frac{32}{3})$

$B = (?, -\frac{32}{3})$: $\Rightarrow -\frac{32}{3} = \frac{2}{3}x^3 - 8x \Rightarrow \underline{-4} = x$ (from hint).

$B = (-4, -\frac{32}{3}).$

$T = 2\int dt$ $\quad C \rightarrow D \rightarrow A.$

$T = \cancel{2\int \frac{dt}{dz} dz}$ $\quad 2\int \frac{dt}{dx}dx = 2\int \frac{dt}{dz}\frac{dz}{dx}dx$

$\frac{dz}{dt} = \frac{1}{\varepsilon}\frac{dy}{dt}$

$= -\frac{x}{\varepsilon}$

$F'(x)$

$C \to D$ is ~~slow~~ v.fast so

$$T \sim 2\int_D^A = 2\int -\frac{\varepsilon}{x}(2x^2-8)\,dx$$

$$= -2\varepsilon\int 2x - \frac{8}{x}\,dx$$

$$= -2\varepsilon\left[x^2 - 8\ln x\right]_4^{2}$$

16ln2

$$= -2\varepsilon\left[4 - 8\ln 2 - 16 + 8\ln 4\right]$$

$$= -2\varepsilon(-12 + 8\ln 2)$$

$$= \underline{\varepsilon(24 - 16\ln 2)}.$$

A = 4 by symmetry

$A = \max(|x(t)|) = 4.$

5 (a) If $I(x) = \int_0^T e^{-xt} f(t)\,dt$

and $f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^n$ as $t \to 0$ $(a_0 \neq 0)$

and $f(t)$ does not grow superexponentially if $T=\infty$ as $t \to$

$$\Rightarrow I(x) \sim \sum_{n=0}^{\infty} \frac{a_n(\alpha+n)!}{x^{n+\alpha-1}}$$

(b) $I = \int_a^b e^{x\phi(t)} f(t)\,dt$. $\phi'(t_0) = 0$. $\phi''(t_0) \neq 0$.

focus on region near the maximum

$$\phi(t) = \phi(t_0) + \cancel{\phi'(t_0)}(t-t_0) + \tfrac{1}{2}\phi''(t_0)(t-t_0)^2 + \dots$$

$$\Rightarrow e^{x\phi(t)} = e^{x\phi(t_0)} e^{\underbrace{-\frac{1}{2}|\phi''(t_0)|(t-t_0)^2 x}_{u^2}}$$

Let $u^2 = -\frac{1}{2}|\phi''(t_0)|(t-t_0)^2 x$ $\Rightarrow 2u\,du = -\frac{1}{2}|\phi''(t_0)|\, 2tx\,dt$

$\Rightarrow dt = \dfrac{-u\,du\,2}{|\phi''(t_0)|\,tx}$

$$\Rightarrow I(x) \sim \int_{-\infty}^{\infty} \frac{e^{x\phi(t_0)} e^{-u^2} f(t)(-u)2}{|\phi''(t_0)|\,tx}\,du$$

$$\sim -\int_{-\infty}^{\infty} \frac{e^{x\phi(t_0)} e^{-u^2} f(t)\,2}{|\phi''(t_0)|\,tx} \left(\frac{|\phi''(t_0)|\,x}{2}\right)^{1/2} (t-t_0)\,du$$

$$= -\int_{-\infty}^{\infty} \frac{e^{x\phi(t_0)} e^{-u^2} f(t)}{t} \left(\frac{2}{x|\phi''(t_0)|}\right)^{1/2} (t-t_0)\,du$$

$$\sim e^{x\phi(t_0)} f(t_0) \sqrt{\frac{2}{x|\phi''(t_0)|}} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$= e^{x\phi(t_0)} f(t_0) \sqrt{\frac{2\pi}{x|\phi''(t_0)|}}$$

5 (a) If $I(x) = \int_0^T e^{-xt} f(t)\, dt$

and $f(t) \sim t^\lambda \sum_{n=0}^{\infty} a_n t^{\lambda_n}$ as $t \to 0$ $\quad \lambda > -1$, $\lambda_0 = 0$

and $f(t)$ does not grow superexponentially if $T = \infty$ as $t \to \infty$

$$\Rightarrow I(x) \sim \sum_{n=0}^{\infty} \frac{a_n (\lambda + \lambda_n)!}{x^{\lambda + \lambda_n + 1}}$$

(b) Focus on the region near the maximum in ϕ and

$$\phi(t) = \phi(t_0) + \cancel{\phi'(t_0)(t-t_0)} + \frac{1}{2}\phi''(t_0)(t-t_0)^2 + \dots$$

$$\Rightarrow e^{x\phi(t)} \sim e^{x\phi(t_0)} e^{-\frac{1}{2}|\phi''(t_0)|(t-t_0)^2 x}$$

Let $u^2 = \frac{1}{2}|\phi''(t_0)|(t-t_0)^2 x$

~~and~~ $2u\, du = \frac{1}{2}|\phi''(t_0)|\, 2(t-t_0) x\, dt$

$$\Rightarrow \quad dt = \frac{[\frac{1}{2}|\phi''(t_0)|(t-t_0)^2 x]^{1/2}}{\frac{1}{2}|\phi''(t_0)|(t-t_0)x}$$

$$= \sqrt{\frac{2}{|\phi''(t_0)|x}}$$

$$\Rightarrow I(x) = \int_{-\infty}^{\infty} e^{x\phi(t_0)} e^{-u^2} f(t_0) \sqrt{\frac{2}{|\phi''(t_0)|}}\, du$$

$$= e^{x(t_0)} \sqrt{\frac{2}{|\phi''(t_0)|}} \int_{-\infty}^{\infty} e^{-u^2}\, du\, f(t_0)$$

$$= e^{x\phi(t_0)} \sqrt{\frac{2\pi}{|\phi''(t_0)|}}\, f(t_0)$$

If $t_0 = a$, $t_0 = b$, answer is half this.

(c) $I_2(x) = \int_{-\pi/2}^{3\pi/2} e^{ix\cos t}\,dt$

$\phi(t) = \cos t$

$a = -\pi/2$

$b = 3\pi/2$

MOSP says $\sim e^{ix\phi(c)} f(c) \sqrt{\frac{2\pi}{|\phi''(c)|x}}\, e^{i\,\mathrm{sgn}(\phi''(c))\pi/4}$

$\phi(t) = \cos t$

$\phi'(t) = -\sin t$ $\qquad \phi'(t) = 0$ at $t = 0, \pi$

$\phi''(t) = -\cos t$

$\Rightarrow \phi''(0) = -1 \rightarrow$ max. with $\phi(0) = 1$

$\phi''(\pi) = 1 \rightarrow$ min. with $\phi(\pi) = -1$.

$\Rightarrow I \sim e^{ix}\sqrt{\frac{2\pi}{x}}\, e^{-i\pi/4} + e^{-ix}\sqrt{\frac{2\pi}{x}}\, e^{i\pi/4}$

Take real part of both sides and using that cos is even $\Rightarrow$

$$\int_{-\pi/2}^{3\pi/2} \cos(x\cos t)\,dt \sim 2\cos\left(x - \frac{\pi}{4}\right)\sqrt{\frac{2\pi}{x}}$$